ON A GENERALIZATION OF
A RESULT OF WINTNER

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SUMMARY

We obtain a generalization of the Hukuwara stability theorem analogous to a recent generalization for second order equations due to Wintner, _Quarterly of Applied Mathematics_, Vol. XV(1958), pp. 428-430.

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In a recent note, [1], Wintner proved the following interesting result.

Theorem 1. *Consider the two equations*

$$\text{(a)} \quad u'' + f(t)u = 0, \qquad \text{(b)} \quad v'' + g(t)v = 0. \tag{1.1}$$

If there exist two linearly independent solutions of (1a), u_1 *and* u_2, *such that*

$$\int^{\infty} (|u_1|^2 + |u_2|^2)|f - g|dt < \infty \tag{1.2}$$

then every solution of (1b) *can be written in the form*

$$v = c_1u_1 + c_2u_2 + O(|u_1| + |u_2|). \tag{1.3}$$

This is an extension of known stability results, cf. [2], to which it reduces if we assume that all solutions of (1a) are bounded as $t \to \infty$.

Let us now show that we can obtain a generalization of this result following the method used in our book, [2], to establish the Hukuwara stability theorem, of which this will be an extension.

Theorem 2. *Consider the vector-matrix systems*

$$\text{(a)} \quad \frac{dx}{dt} = A(t)x, \qquad \text{(b)} \quad \frac{dy}{dt} = B(t)y. \tag{1.4}$$

Let $X(t)$ *be the solution of*

$$\frac{dX}{dt} = A(t)X, \quad X(0) = I. \tag{1.5}$$

<u>If</u>

$$\int^{\infty} ||B(t) - A(t)|| \; ||X(t)|| \; ||X^{-1}(t)||dt < \infty , \tag{1.6}$$

<u>then every solution of</u> (1b) <u>may be written</u>

$$y = Xc + O(||X||) \tag{1.7}$$

<u>as</u> $t \to \infty$.

The norms of matrices and vectors are taken to be respectively $\sum_{i,j} |x_{ij}|$ and $\sum_i |x_i|$.

<u>Proof</u>. Write

$$\frac{dy}{dt} = A(t)y + (B(t) - A(t))y. \tag{1.8}$$

Then, if $y(0) = b$, we have

$$y = X(t)b + \int_0^t X(t)X^{-1}(s)(B(s) - A(s))yds. \tag{1.9}$$

Hence

$$||y|| \leq ||X(t)|| \; ||b|| + \int_0^t ||X(t)|| \; ||X^{-1}(s)|| \; ||B(s) - A(s)|| \; ||y||ds. \tag{1.10}$$

Thus, if we set

$$\begin{aligned} u(t) &= ||X^{-1}(t)|| \; ||B(t) - A(t)|| \; ||y(t)||, \\ v(t) &= ||B(t) - A(t)|| \; ||X(t)|| \; ||X^{-1}(t)||, \end{aligned} \tag{1.11}$$

we obtain the scalar inequality

$$u \leq c_1 v + v \int_0^t u\,ds. \tag{1.12}$$

This yields, as a consequence of the fundamental inequality, [2], or directly, the estimate

$$\int_0^t u\,ds \leq c_1 \int_0^t v(s) e^{\int_s^t v\,dr}\,ds. \tag{1.13}$$

By assumption, $\int^{\infty} v\,ds < \infty$. Hence, the integral

$$\int_0^{\infty} X^{-1}(s)(B(s) - A(s))y\,ds \tag{1.14}$$

converges. This means that we can write (1.9) in the form

$$\begin{aligned} y = X(t)b + X(t)\int_0^{\infty} X^{-1}(s)(B(s) - A(s))y\,ds \\ - X(t)\int_t^{\infty} X^{-1}(s)(B(s) - A(s))y\,ds, \end{aligned} \tag{1.15}$$

which yields the stated result.

REFERENCES

1. Wintner, A., "On Linear Perturbations," Quart. Appl. Math., Vol. XV(1958), pp. 428-430.

2. Bellman, R., Stability Theory of Differential Equations, McGraw-Hill Book Company, Inc., New York, 1954.